* A region is bounded if it lies in a disk of finite radius.
* A region is unbounded if it is not bounded
* A point (x_0, y_0) is an interior point if it is the center of a disk that lies entirely in the domain
* A point (x_0, y_0) is a boundary if the disk centered at (x_0, y_0) contain points outside the domain
* A region is closed if it contains all its boundary points
* The set of points in which a function $f(x,y) = c$ is called a level curve of f
* a function $f(x,y,z) = c$ is called a level surface
* we say

$$\lim_{(x,y)\to(0,0)} f(x,y) = L$$

if $\forall\, \epsilon > 0,\ \exists\, \delta > 0$ such that $\forall x\ \forall y$ in D_f

we have

$$|f(x,y) - L| < \epsilon \quad \text{whenever } \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

* $f(x,y)$ is continuous at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$$

such that $f(x_0, y_0)$ is defined in D_f

and $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

* if $f(x,y)$ has 2 different limits along two different paths, at the same point, this limit does not exist
* Definition of a partial derivative

$$\left.\frac{\partial f}{\partial x}\right|_{x_0,y_0} = f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+h, y) - f(x_0, y_0)}{h}$$

$$\left.\frac{\partial f}{\partial y}\right|_{x_0,y_0} = f_y(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

* Mixed Derivative: if $f(x,y)$, f_x, f_y, f_{yx} and f_{xy} are defined throughout an open region at (a,b) and are all continuous at (a,b), then:

$$f_{xy}(a,b) = f_{yx}(a,b)$$

* A function $z = f(x,y)$ is differentiable at (x_0, y_0) if $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist and satisfy the equation:

$$\Delta z = f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta z + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$.

or

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - f_x(x_0,y_0)\,\Delta x - f_y(x_0,y_0)\,\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} \text{ exists}$$

* f is differentiable if $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous throughout the open region.

* if $w = f(x(t), y(t))$ [CHAIN RULE]

then $$\frac{dw}{dt} = \frac{\partial f}{\partial x} \times \frac{dx}{dt} + \frac{\partial f}{\partial y} \times \frac{dy}{dt}$$

if $w = f(x(t), y(t), z(t))$

then $$\frac{dw}{dt} = \frac{\partial f}{\partial x} \times \frac{dx}{dt} + \frac{\partial f}{\partial y} \times \frac{dy}{dt} + \frac{\partial f}{\partial z} \times \frac{dz}{dt}$$

if $w = f(x,y)$ and $x = g(r,s)$ $y = u(r,s)$

then $$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial s}$$

* Directional Derivative:

$$\left(\frac{df}{ds}\right)_{\vec{u}, P_0} = \left(D_{\vec{u}} f\right)_{P_0} = \lim_{s\to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

such that $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$ is a unit vector

* $\vec{\nabla} f = f_x \vec{i} + f_y \vec{j}$ if $f(x,y)$

$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$ if $f(x,y,z)$

* $$\left(\frac{df}{ds}\right)_{\vec{u}, P_0} = \left(\vec{\nabla} f\right)_{P_0} \cdot \vec{u}$$ [$\vec{u}$ IS A UNIT VECTOR]

* $D_{\vec{u}}f = \vec{\nabla}f \cdot \vec{u}$

$$= |\vec{\nabla}f| \cdot |\vec{u}| \cdot \cos\theta \qquad (\theta \text{ is the angle between } \nabla f \text{ and } u)$$

$$= |\vec{\nabla}f| \cdot \cos\theta$$

f increases most rapidly when: (in the direction ∇f)

$$D_{\vec{u}}f = |\nabla f| \cos(0) = |\nabla f|$$

f decreases most rapidly when: (in the direction ∇f)

$$D_{\vec{u}}f = |\nabla f| \cos(\pi) = -|\nabla f|$$

* Tangent to level curve

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$$

* $\vec{\nabla}(f+g) = \vec{\nabla}f + \vec{\nabla}g$
* $\vec{\nabla}(f-g) = \vec{\nabla}f - \vec{\nabla}g$
* $\vec{\nabla}(kf) = k\vec{\nabla}f$ (k is a number)
* $\vec{\nabla}(fg) = f\vec{\nabla}g + g\vec{\nabla}f$
* $\vec{\nabla}\left(\frac{f}{g}\right) = \dfrac{g\vec{\nabla}f - f\vec{\nabla}g}{g^2}$

* Tangent plane:

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

* normal line:

$$\begin{cases} x = x_0 + f_x(P_0)\, t \\ y = y_0 + f_y(P_0)\, t \\ z = z_0 + f_z(P_0)\, t \end{cases} \qquad t \in \mathbb{R}$$

* $df = D_{\vec{u}}f \cdot \underbrace{ds}_{\text{increment}}$

* $L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

$$f(x,y) \simeq L(x,y)$$